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Superposition, Entanglement, and Product of States

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The superposition relation extended to the statistical operators is shown to be invariant under tensor product and partial trace operations. Particular mathematical examples of superposition are characterized as well as the nature of the Schmidt decomposition of pure states superposition of other pure states.

KEY WORDS: superposition; entanglement; product.

1. INTRODUCTION

Entanglement of states is a point where the difference between Quantum Mechanics and Classical Mechanics is more evident. Entanglement of pure states is a concept directly connected to the foundations of Quantum Mechanics, being a pure superposition of products of pure states, a possibility that exists only trivially in Classical Mechanics.

The importance of the concept of entanglement of states became evident after the critical discussion on the completeness of Quantum Mechanics in the famous paper by Einstein *et al.* (1935). The notion of entanglement is important also in relation with the results by Bell *et al.* (2001). (See, e.g., Peres, 1995.) By using the Schmidt decomposition of nonfactorable entangled states of two quantum systems it is possible to violate (Gisin and Peres, 1992) Bell's inequality in the form of the CHSH inequality (Clauser *et al.*, 1969) in agreement with the experimental results (Aspect *et al.*, 1982).

Entanglement plays also, in a direct or an indirect way, an important role in subsequent fields of research such as quantum computing and quantum communication (Colin, 1999) and quantum teleportation (Bennet *et al.*, 1993).

The notion has been extended from the pure states to the density operators for compound systems. An entangled density operator is a nonseparable density

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operator, namely an operator that cannot be expressed by a diagonal sum of products of density operators with positive coefficients. Necessary and sufficient conditions have been given for the separability of the states (Horodecki *et al.*, 1996), and, due to their interest, entanglement measures have also been defined (e.g., Rudolph, 2001). On the analogy of the pure states case, the extended notion of entanglement can be compared with the extended notion of superposition for the density operators (Zecca, 1980). This definition which is not very used in the literature, is an application to the density operators of a definition that holds at the level of a general quantum logic and that has been already used by Varadarajan (1968). (For a review about the superposition relation one can refer to Zecca, 1981.)

The object of the present paper is that of studying, in an elementary way, the relation among superposition, entanglement, and product of statistical operators in some particular mathematical situations. The study is done in the context of the Hilbert model. After recalling some properties of the standard logic, the definition of superposition is formulated, in equivalent ways, in that language and exemplified for different forms of statistical operators. Some preliminary results are then given and some aspects of the behavior of superposition under tensor product are put into evidence. The structure of the states under particular superposition relations is then studied and characterized in terms of their separability properties, Schmidt decomposition and reduction by partial trace operations. Even if the study is performed elementally and the cases considered are relatively simple, the results seem to be sufficiently indicative.

2. SUPERPOSITION OF STATES IN STANDARD LOGIC: DEFINITIONS AND PRELIMINARY RESULTS

It is usefull to recall some aspects of the so called standard logic approach to Quantum Mechanics. This is an axiomatic description of Quantum Mechanics originated from a paper by Birkhoff and von Neumannn (1936). [A general formulation and the Hilbert model realization from the propositional calculus can be found in Jauch (1968) and Piron (1976). For successive different developments, one can refer to Beltrametti and Cassinelli (1981).] According to that approach, to the irreducible quantum physical system Σ there is associated a pair L, S where $L \equiv L(H)$ is the complete orthomodular atomic lattice of the closed subspaces (propositions) of a separable complex Hilbert space of dimension ≥ 3. The set S is the set of all σ-additive probability measures (states) on $L(H)$. A proposition a represents a class of equivalent yes–no experiments on Σ and a state s a preparing procedure of Σ. The number $s(a)$ gives the probability of the outcome yes for a test of a when Σ is prepared according to the state s. By Gleason's theorem (Gleason, 1957), for every state s there is one and only one $\rho \in K(H)$ such that

$$s(a) = s_\rho(a) = Tr\rho P^a, \quad a \in L(H) \tag{1}$$

$K(H)$ being the set of positive trace class operators of trace 1 of H (density operators) (Schatten, 1960) and P^a the orthogonal projection of range a. In the following we make the identification $S \equiv K(H)$ and write $s_\rho(a) = \rho(a)$. In case a is a one-dimensional subspace of H we write also $P^a = P^\psi$ and $a = [\psi]$, where ψ is a unit vector of a.

Definition 1. (Varadarajan, 1968). A state ρ is superposition of the states of $D \subset K(H)$ if anyone of the following conditions hold:

$$\sigma(a) = 0\,\forall\sigma \in D, \quad a \in L(H) \Rightarrow \rho(a) = 0 \tag{2}$$

$$\sigma(b) = 1\,\forall\sigma \in D, \quad b \in L(H) \Rightarrow \rho(b) = 1 \tag{3}$$

The formulations (2), (3) are indeed equivalent because, from the additivity of the states, $\rho(a) = 0$ iff $\rho(a^\perp) = 1$, $a^\perp$ being the Hilbertian orthogonal complement of a in $L(H)$. By considering the spectral decomposition of a density operator and the representation theorem (1), it is not difficult to show that ρ is superposition of the states in D, in the sense of (2) or (3), if and only if

$$[\rho] \leq \bigvee_{\sigma\in D} [\sigma] \tag{4}$$

where $[\rho]$ denotes the range of ρ as an operator in H; $\leq$ denotes set theoretical inclusion; and $\vee_{\sigma\in D}[\sigma]$ the closure of the linear span of the subspaces $[\sigma]$, $\sigma \in D$ of H. (For a formal proof see, e.g., Gorini and Zecca, 1975; Zecca, 1980.) In case of the pure states $\rho = P^\psi$ and $D = \{P^{\psi_1}, P^{\psi_2}\}$, the result (4) implies easily

$$\psi = \alpha\psi_1 + \beta\psi_2 \quad (|\alpha|^2 + |\beta|^2 = 1) \tag{5}$$

whose content is that of superposition of pure states in Dirac's sense (Dirac, 1947). If instead ρ is a convex combination $\rho = \Sigma_i \alpha_i \rho_i$, $(\rho, \rho_i \in K(H))$, then ρ is superposition of the set of states $\{\rho_i\}$ for which is holds $[\rho] = \vee_i[\rho_i]$.

Suppose now we have a second phisical system $\tilde{\Sigma}$ with associated logic and states $L(\tilde{H})$, $K(\tilde{H})$. According to the standard formulation of quantum mechanics, the compound system $\Sigma + \tilde{\Sigma}$ has logic $L(H \otimes \tilde{H})$ and states $K(H \otimes \tilde{H})$. At the level of abstract proposition-state structure a notion of product is possible, which preserves superposition (Zecca, 1994). It is a fact that this property does hold in the Hilbert model.

Proposition 1. *Let $\rho \in K(H)$, $D \subset K(H)$ and $\tilde{\rho} \in K(H)$, $D \subset K(\tilde{H})$. Then the following are equivalent.*

(i) *ρ is superposition of the states of D and $\tilde{\rho}$ is superposition of the states of $\tilde{D}$;*

(ii) *$\rho \otimes \tilde{\rho}$ is superposition of the set of states $D \otimes \tilde{D} = \{\sigma \otimes \tilde{\sigma} : \sigma \in D, \tilde{\sigma} \in \tilde{D}\}$.*

Proof: In $L(H \otimes \tilde{H})$ one has that $[\rho] \otimes [\tilde{\rho}] \leq \vee_{\sigma \in D} \otimes \vee_{\tilde{\sigma} \in \tilde{D}}$ if and only if both $[\rho] \leq \vee_{\sigma \in D}[\sigma]$ and $[\tilde{\rho}] \leq \vee_{\tilde{\sigma} \in D}[\tilde{\sigma}]$ (compare with Zecca, 1994). The proof is completed by taking into account that it also holds $[\rho] \otimes [\tilde{\rho}] = [\rho \otimes \tilde{\rho}]$ and $\vee_{\sigma \in D} \otimes \vee_{\tilde{\sigma} \in \tilde{D}}[\tilde{\sigma}] \equiv \vee_{\sigma \in D, \tilde{\sigma} \in \tilde{D}}[\sigma \otimes \tilde{\sigma}]$. □

Suppose now ρ is a separable state of $K(H \otimes \tilde{H})$, that is $\rho = \Sigma_i \omega_i \rho_i \otimes \tilde{\rho}_i$ (e.g., Rudolph, 2001) (ω_i positive numbers). Then also here ρ is a special superposition of the set of states $\{\rho_i \otimes \tilde{\rho}_i\}$ for which it holds $[\rho] = \vee_i [\rho_i] \otimes [\tilde{\rho}_i]$. There follows that every pure state (one-dimensional projection) associated to the spectral decomposition of ρ is a superposition, not only of ρ, but also of the family of states $\{\rho_i \otimes \tilde{\rho}_i\}$.

3. PARTICULAR CONFIGURATIONS OF SUPERPOSITION

It is clear from the previous considerations that there are relevant superpositions with respect to family of pure states of the form $\{\varphi_i \otimes \tilde{\varphi}_k\}$, i, k in some index set I. A characterization of some of these situations are now given.

Proposition 2. *Let ρ be a superposition of the set of pure states $\{P^{\varphi_i} \otimes P^{\tilde{\varphi}_k}, i, k = 1, 2\}$ with $\{\varphi_i\} \subset H$, $\{\tilde{\varphi}_k\} \subset \tilde{H}$. Then $\rho = P^{\varphi_1} \otimes \tilde{\rho}$ for some $\tilde{\rho}$ if and only if $\varphi_2 = \exp(i\lambda)\varphi_1$, λ, real.*

Proof: The spectral decomposition of ρ has the form

$$\rho = \sum_{k=1}^{\dim[\rho]} \alpha_k P^{\eta k}$$

$(\eta_i|\eta_k) = \delta_{ik}$ in $H \otimes \tilde{H}$. By the superposition assumption one has also the representation $\eta_j = \Sigma_{ik}\alpha_{ik}^j \varphi_i \otimes \tilde{\varphi}_k$. Suppose now $\varphi_2 = \exp(i\lambda)\varphi_1$. Then obviously, η_j is of the form $\eta_j = \varphi_1 \otimes \tilde{\chi}_j$ and ρ is of the form $\rho = \Sigma_k \alpha_k P^{\varphi_1 \otimes \tilde{\chi}_k} = P^{\varphi_1} \otimes \tilde{\rho}$.

Conversely suppose $\rho = P^{\varphi_1} \otimes \tilde{\rho}$. By considering the spectral decomposition of $\tilde{\rho}$, ρ is of the form $\rho = \Sigma_k \beta_k P^{\varphi_1 \otimes \tilde{\eta}_k}$, $(\tilde{\eta}_i|\tilde{\eta}_k) = \delta_{ik}$ in H. Hence $[\rho] = [\varphi_i] \otimes \vee_k [\tilde{\eta}_k]$, the index k ranging from 1 to $\dim[\rho]$. From the superposition assumption and relation (1), any vector in $[\rho]$ has both the forms

$$\varphi_i \otimes \sum_{j}^{\dim[\rho]} \beta_j \tilde{\eta}_j = \sum_{i,k=1}^{2} \alpha_{ik} \varphi_i \otimes \tilde{\varphi}_k.$$

By projecting the last equation onto φ_i and φ_2 in H one gets respectively

$$\beta_j \tilde{\eta}_j = \alpha_{ik}(\varphi_1|\varphi_i)\tilde{\varphi}_k; \quad (\varphi_2|\varphi_1)\beta_j \tilde{\eta}_j = \alpha_{ik}(\varphi_2|\varphi_i)\tilde{\varphi}_k$$

where repeated indexes are understood to be summed. This implies $\alpha_{ik}(\varphi_1|\varphi_i)(\varphi_2|\varphi_1) = \alpha_{ik}(\varphi_2|\varphi_i)$, $k = 1, 2$. By developing the sum one gets $|(\varphi_2|\varphi_1)| = 1$ and hence the result since $\|\varphi_1\| = \|\varphi_2\| = 1$.

The previous result holds also for $n > 2$ by strengthening the superposition assumption. □

Proposition 3. *Let $D = \{P^{\varphi_i} \otimes P^{\tilde{\varphi}_k}, i, k, = 1, 2, \ldots, n\}$, $n > 2$, $\{\varphi_i\}$, $\{\tilde{\varphi}_k\}$ unit vectors in H, $\tilde{H}$ respectively. Then the following conditions are equivalent*

(i) *any superposition ρ of D is to the form $\rho = P^{\varphi_1} \otimes \tilde{\rho}$*
(ii) $\varphi_k = \exp(i\lambda_k)\varphi_1$ $(\lambda_k \in \Re)$, $k = 1, 2, \ldots, n$.

Proof: Condition (ii) implies condition (i) in the same elementary way as in Proposition 2. To show the converse, note that any $\psi \in [\rho]$ can be simultaneously written as

$$\varphi_1 \otimes \sum_{k=1}^{\dim[\rho]} \alpha_k \tilde{\eta} k = \sum_{i,k=1}^{2} \alpha_{ik} \varphi_i \otimes \tilde{\varphi}_k$$

Since this holds for any superposition of D, the last equation holds also by choosing $\alpha_{ik} = 0$ for $i \neq 1, i_1, k \neq 1, k_1$. By proceeding then as in the previous Proposition one gets $|(\varphi_{i_1}|\varphi_1)| = 1$ and, the index i_1 being arbitrary, there follows the conclusion (ii). □

[Results similar to those of Propositions 2,3 hold obviously also for the other component of the tensor product.] For what concerns the relation between superposition and Schmidt representation (Ekert and Knight, 1995; Schmidt, 1906 and references therein) one has the following.

Proposition 4. *Let P^{ψ} be superposition of the states in $\{P^{\varphi_i} \otimes P^{\tilde{\varphi}k}, i, k, = 1, 2, \ldots, n\}$, $n > 2$, $\{\varphi_i\}$, $\{\tilde{\varphi}_k\}$ linearly independent unit vectors in H, $\tilde{H}$ respectively. By (1), $\psi = \Sigma_{ik}\alpha_{ik}\varphi_i \otimes \tilde{\varphi}_k$ and by the Schmidt representation $\psi = \Sigma_l p_l u_l \otimes \tilde{u}_l$. Then the following properties hold:*

(i) $u_i \in \vee_k[\varphi_k]$, $\tilde{u}_i \in \vee[\tilde{\varphi}_k]$, $(i = 1, \ldots, n)$ *and* $p_l = 0, l > n$.
(ii) $p_l \neq 0, l = 1, 2, \ldots, n \Leftrightarrow \det \alpha_{ik} \neq 0$.

Proof: After projecting the identity $\Sigma_l p_l u_l \otimes \tilde{u}_l = \Sigma_{il}^n \alpha_{il}\varphi_i \otimes \tilde{\varphi}_l$ onto φ_k in H and onto $\tilde{\varphi}_k$ in $\tilde{H}$ one obtains (no sum over k here)

$$p_k u_k = \sum_{il}^{n} \alpha_{il}(u_k|\varphi_l)\tilde{\varphi}_l; \quad p_k \tilde{u}_k = \sum_{il}^{n} \alpha_{il}(\tilde{u}_k|\tilde{\varphi}_l)\tilde{\varphi}_l$$

There follows that, by suitably renumbering the index, one has $p_l = 0$ for $l > n$. Let now $\{v_i\}$, $\{\tilde{v}_i\}$ be orthonormal bases in $\vee_i[\varphi_i]$, $\vee_i[\tilde{\varphi}_i]$ respectively, so choosen that

$$v_i = u_i, \quad \tilde{v}_i = \tilde{u}_i, \quad i \text{ such that } p_i \neq 0$$

Then $\varphi_i = \Sigma_k A_{ik} v_k$, $\tilde{\varphi}_j = \Sigma_l B_{jl} \tilde{v}_l$, where the matrices A, B are non singular. It is then possible to write $\Sigma_l p_l u_l \times \tilde{u}_l = \Sigma_l^n p'_l v_l \otimes \tilde{v}_l$ by defining $p'_l = p_l$ for $p_l \neq 0$ and $p'_l = 0$ otherwise. Hence

$$\begin{aligned}\sum_l^n p'_l v_l \otimes \tilde{v}_l &= \sum_{ik} \alpha_{ik} \varphi_i \otimes \tilde{\varphi}_k \\ &= \sum_{ikml} \alpha_{ik} A_{il} B_{km} v_l \otimes \tilde{v}_m\end{aligned}$$

Therefore $\Sigma_{ik} A^\top_{li} \alpha_{ik} B_{km} = p'_l \delta_{lm}$. The result (ii) follows by taking the determinant: $\det A \det B \det \alpha = \Pi_{l=1}^n p'_l$. □

A special case of Proposition 4 is that in which $(\varphi_i|\varphi_k) = \delta_{ik}$, $(\tilde{\varphi}_i|\tilde{\varphi}_k) = \delta_{ik}$. Then A, B are unitary matrices and $\det \alpha = \pm \Pi_l p'_l$.

Also the operation of taking partial trace is compatible with superposition. If $\{\varphi_h\}$, $\{\tilde{\varphi}_k\}$ are complete orthonormal systems in H, $\tilde{H}$ the partial trace operators of a density operator $\sigma \in K(H \otimes \tilde{H})$ will be denoted $\sigma_\varphi = Tr_{\tilde{\varphi}} \sigma = \Sigma_k \langle \tilde{\varphi}_k | \sigma | \tilde{\varphi}_k \rangle$ and $\sigma_{\tilde{\varphi}} = Tr_\varphi \sigma = \Sigma_i \langle \varphi_i | \sigma | \varphi_i \rangle$. □

Proposition 5. *Let σ be superposition of the states of $D = \{\rho^\alpha; \alpha \in I\} \subset K(H \otimes \tilde{H})$. Then σ_φ is superposition of the states of $D_\varphi = \{\rho_\varphi : \rho \in D\}$ and $\sigma_{\tilde{\varphi}}$ is superposition of the states of $D_{\tilde{\varphi}} = \{\rho_{\tilde{\varphi}} : \rho \in D\}$ for any $\{\varphi_h\}$, $\{\tilde{\varphi}_k\}$ complete orthonormal systems in H, $\tilde{H}$.*

Proof: From the spectral decomposition of a density operator one has, for every α,

$$\rho^\alpha = \sum_i \beta_i^\alpha P^{\phi_i^\alpha}, \qquad \phi_i^\alpha \perp \phi_k^\alpha, \quad i \neq k$$

where, from the completeness of $\{\varphi_i \otimes \tilde{\varphi}_k\}$, $\phi_i^\alpha = \Sigma_{ab} C_{iab}^\alpha \varphi_a \otimes \tilde{\varphi}_b$. Hence

$$\rho^\alpha = \sum_{iablm} \beta_i^\alpha C_{iab}^\alpha C_{ilm}^{\alpha *} |\varphi_a\rangle\langle\varphi_l| \otimes |\tilde{\varphi}_b\rangle\langle\tilde{\varphi}_m|$$

Therefore

$$\rho_\varphi^\alpha = \sum_{iabl} \beta_i^\alpha C_{iab} C_{ilb}^* |\varphi_a\rangle\langle\varphi_l|$$

We now show the result by applying the Definition (2), namely by showing that $\rho_\varphi^\alpha(x) = 0 \,\forall\alpha \Rightarrow \sigma_\varphi(x) = 0$, $x \in L(H \otimes \tilde{H})$. One first gets from the last equation, by writing P^x in terms of one-dimensional projections, $P^x = \Sigma_k |x_k\rangle\langle x_k| (x_i \perp x_k, i \neq k)$,

$$0 = \rho_\varphi^\alpha(x) = \sum_{hib} \beta_i^\alpha \left(\sum_a C_{iab}^\alpha \langle x_h|\varphi_a\rangle \right) \left(\sum_l C_{ilb}^{\alpha *} \langle \varphi_l|x_h\rangle \right)$$

that implies

$$\sum_a C_{iab}^\alpha \langle x_h|\varphi_a\rangle = 0 \; \forall \alpha, i, b, h \tag{6}$$

Consider now the spectral decomposition of $\sigma = \Sigma_i \gamma_i P^{\eta_i}$. One has, by assumptions, $[\sigma] \leq \vee_\alpha [\rho^\alpha] = \vee_{\alpha i} [\phi_i^\alpha]$. Therefore η_i can be represented as $\eta_i = \Sigma_{k\alpha'} A_{ik}^{\alpha'} \phi_k^{\alpha'}$, where the sum over α' is at most countable. There follows

$$\sigma = \sum_{\alpha'\beta' ijk} \gamma_i A_{ik}^{\alpha'} A_{ij}^{\beta' *} \left|\phi_k^{\alpha'}\right\rangle\!\left\langle\phi_j^{\beta'}\right|$$

By using the representation of the $\phi's$ in terms of the $\varphi_i \otimes \tilde{\varphi}_k$'s one has also

$$\sigma_\varphi = \sum_l \langle \tilde{\varphi}_l|\sigma|\tilde{\varphi}_l\rangle = \sum_{\alpha'\beta' ijkabc} \gamma_i A_{ik}^{\alpha'} A_{ij}^{\beta' *} C_{kab}^{\alpha'} C_{jcb}^{\beta' *} |\varphi_a\rangle\langle\varphi_c|$$

Therefore

$$\sigma_\varphi(x) = \sum_{\alpha'\beta' ijkabch} \gamma_i A_{ik}^{\alpha'} A_{ij}^{\beta' *} C_{kab}^{\alpha'} C_{jcb}^{\beta' *} \langle\varphi_c|x_h\rangle\langle x_h|\varphi_a\rangle = 0$$

by using the result in (6). The second part of the proof can be performed in a completely similar way. □

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